*International Journal of Theoretical Physics, Vol. 39, No. 2, 2000*

# Statistical Mechanical and Thermodynamic Entropies of the Einstein–Maxwell Dilaton–Axion Black Hole

**Jiliang Jing[1,2] and Mu-Lin Yan[2]**

*Received May 26, 1999*

We discuss the connection between the statistical mechanical and thermodynamic entropies due to the nonminimally coupled scalar fields on the Einstein–Maxwell dilaton–axion black hole spacetime. It is demonstrated that although the statistical mechanical entropy and one-loop correction to the thermodynamic entropy are equivalent for coupling $\xi \leq 0$, the presence of the bare pure geometrical contribution excludes the possibility to identify the statistical mechanical entropy with the thermodynamic entropy if we use the standard renormalization scheme.

## 1. INTRODUCTION

The study of the relation between the statistical mechanical and thermodynamic entropies has attracted much attention [1–6] motivated by attempts to explain the entropy of black holes as the statistical mechanical entropy of quantum fields propagating near the event horizon. The thermodynamic entropy of a black hole is related to the covariant Euclidean free energy $F^E$, which can be calculated by the method of the conical singularity [1]. This procedure has been consistently carried out for some black holes [7–16]. The statistical mechanical entropy can be derived from the canonical formulation $F^C$ [1]. One way to calculate the $F^C$ is by the "brick wall model" (BWM) proposed by 't Hooft [17]. He argued that the black hole entropy is identified with the statistical mechanical entropy arising from a

[1] Physics Department and Institute of Physics, Hunan Normal University, Changsha, Hunan 410081, China.
[2] Department of Astronomy and Applied Physics, University of Science and Technology of China, Hefei, Anhui 230026, China.

thermal bath of quantum fields propagating outside the horizon. In the model, in order to eliminate divergence which appears due to the infinite growth of the density of states close to the horizon, 't Hooft introduces a "brick wall" cutoff: a fixed boundary near the event horizon within the quantum field does not propagate and the Dirichlet boundary condition is imposed on the boundary [wave function $\phi = 0$ for $r = r(\Sigma_h)$]. The model has been successfully used in studies of the entropy for some black holes [17–28]. Recently, in order to get the correct $\xi$ dependence for the statistical entropy of the black hole, Solodukhin [3] proposed modification of the original 't Hooft "brick wall" prescription in which, instead of the Dirichlet condition, he suggested some scattering ansatz for the field functions at the horizon, i.e., the field function near the horizon that describes scattering by the hole with some nontrivial change of phase.

Solodukhin [3] calculated the statistical mechanical entropy for a scalar field with nonminimal coupling on the static black hole spacetime and found that for $\xi \leq 0$ the result agrees with the one-loop correction to the thermodynamic entropy [11]. Frolov and Fursaev [1] review the study of the relation between the thermodynamic entropy and the statistical mechanical entropy of black holes and showed that the covariant Euclidean free energy $F^E$ and the statistical mechanical free energy $F^C$ are equivalent when ones uses ultraviolet regularization [1] for static black holes.

Research on rotating black holes has also attracted much interest. Mann and Solodukhin [16] studied the thermodynamic entropy due to a minimally coupled scalar field by using the conical singularities method. Lee and Kim [20, 25], using the 't Hooft BWM, discussed the statistical mechanical entropy of the Kerr black hole, the Kerr–Newman black hole, and the Kaluza–Klein black hole. However, as Mann and Solodukhin showed [16], the calculations performed in refs. 20 and 25 appear to be unsatisfactory. Frolov and Fursaev [1] pointed out that for the stationary axisymmetric black hole, the relation between the two entropies has not been investigated yet.

The aim of this paper to compare the statistical mechanical entropy of the Einstein–Maxwell dilaton–axion (EMDA) black hole obtained by using the 't Hooft BWM and regularized by the Pauli–Villars scheme with its thermodynamic entropy by means of conical singularities.

The paper is organized as follows: In Section 2 the spacetime of the EMDA black hole is introduced. In Section 3, 't Hooft's BWM [3] is used to calculate the statistical mechanical entropy for nonminimally coupled scalar fields in the EMDA black hole. In Section 4 the thermodynamic entropy is studied by means of the conical singularities method. A summary is presented in the last section.

## 2. STATIONARY AXISYMMETRIC EMDA BLACK HOLE

The stationary axisymmetric EMDA black hole solution (we take the solution $b = 0$ in Eq. (14) in ref. 2 because the solution $b \neq 0$ cannot be interpreted properly as a black hole) is given by [29]

$$\begin{aligned} ds^2 = &-\frac{\Sigma - a^2\sin^2\theta}{\Delta}\,dt^2 + \frac{\Delta}{\Sigma}\,dr^2 + \Delta d\theta^2 + \frac{\sin^2\theta}{\Delta}\,[(r^2 + a^2 - 2Dr)^2 \\ &- \Sigma a^2 \sin^2\theta]\,d\varphi^2 - \frac{2a\sin^2\theta}{\Delta}\,[(r^2 + a^2 - 2Dr)^2 - \Sigma]\,dt\,d\varphi \\ = &\, g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\varphi\varphi}\,d\varphi^2 + 2g_{t\varphi}\,dt\,d\varphi \end{aligned} \tag{2.1}$$

with

$$\Sigma = r^2 - 2mr + a^2, \qquad \Delta = r^2 - 2Dr + a^2\cos^2\theta$$

$$e^{2\Phi} = \frac{W}{\Delta} = \frac{\omega}{\Delta}(r^2 + a^2\cos^2\theta), \qquad \omega = e^{2\Phi_0}$$

$$K_a = K_0 + \frac{2aD\cos\theta}{W}, \qquad A_t = \frac{1}{\Delta}(Qr - ga\cos\theta)$$

$$A_r = A_\theta = 0, \qquad A_\varphi = \frac{1}{a\Delta}[-Qra^2\sin^2\theta + g(r^2 + a^2)a\cos\theta]$$

where $A_\mu$ is the electromagnetic potential, $\Phi$ is the massless dilaton field, and $K_a$ is the axion field dual to the three-index antisymmetric tensor field $H = -e^{-\Phi}dK_a/4$. The mass $M$, angular momentum $J$, electric charge $Q$, and magnetic charge $P$ of the black hole are, respectively,

$$M = m - D, \qquad J = a(m - D), \qquad Q = \sqrt{2\omega\,D(D - m)}, \qquad P = g \tag{2.2}$$

For the black hole, there is a unique time-translational Killing vector $\xi^\mu(t) = (1, 0, 0, 0)$ and a unique rotational Killing vector $\xi^\mu(\varphi) = (0, 0, 0, 1)$. From the formula of the surface gravity $\kappa^2 = \lim[-(\chi^b \nabla_b \chi^c)(\chi^a \nabla_a \chi_c)\chi^d \chi_d]$, where $\chi^a = \xi^a(t) + \Omega_H\xi^a(\varphi)$ [30, 31], we obtain

$$\begin{aligned} \kappa &= \frac{-1}{2}\lim_{r\to r_+}\left(\sqrt{\frac{-1}{g_{rr}(g_{tt} - g_{t\varphi}^2/g_{\varphi\varphi})}}\,\frac{d}{dr}\left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\right) \\ &= \frac{r_+ - r_-}{2(r_+^2 - 2Dr_+ + a^2)} = \frac{2\pi}{\beta_H} \end{aligned} \tag{2.3}$$

The above discussion shows that the EMDA black hole has several different properties compared to the Kerr–Newman black hole: (a) Two horizons of the Kerr–Newman black hole are give by $r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$, whereas for the EMDA black hole

$$r_\pm = \left(M - \frac{Q^2}{2\omega M}\right) \pm \sqrt{\left(M - \frac{Q^2}{2\omega M}\right)^2 - a^2} \tag{2.4}$$

(b) The Kerr–Newman metric has singularities at $r^2 + a^2 \cos^2\theta = 0$, whereas the EMDA black hole has singularities at $r^2 - 2Dr + a^2 \cos^2\theta = 0$. On other hand, the Kerr black hole, the static Garfinkle–Horowitz–Strominger dilatonic black hole, and the Schwarzschild black hole are special cases of the EMDA black hole.

## 3. STATISTICAL MECHANICAL ENTROPY OF THE EMDA BLACK HOLE

When we assume that the scalar field in the EMDA black hole is rotating with an azimuthal angular velocity $\Omega_0$; then the partition function can be written as $Z^q = \Sigma_{n_{q,m}} \exp[-\beta(E_q - \Omega_0 m)n_q]$, where $q$ denotes the quantum state of the field with energy $E_q$ and angular momentum $m$. The free energy is given by

$$F^q = \frac{1}{\beta} \int dm \int dp_\theta \int_0^\infty dn(E, m, p_0) \ln\{1 - \exp[-\beta(E - \Omega_0 m)]\} \tag{3.1}$$

We now first seek the quantization condition by using the motion equation of the scalar field and introducing Solodukhin's boundary condition, and then using the quantization condition to calculate the Helmholtz free energy (3.1).

Using the WKB approximation with $\phi = \exp[-iEt + im\phi + iW(r, \theta)] = \exp[-iEt + im\phi]\psi(r, \theta)$, and substituting the EMDA black hole metric (2.1) into the equation of the scalar field with mass $\mu$ and arbitrary coupling to the scalar curvature $R$, we have [21]

$$\begin{aligned}(\partial_r W(r, \theta))^2 = \frac{1}{g^{rr}} [&-g^{tt}E^2 + 2g^{t\varphi}Em - g^{\varphi\varphi}m^2 - g^{\theta\theta}mp_\theta^2 \\ &- (\mu^2 + \xi R)]\end{aligned} \tag{3.2}$$

where $p_0 \equiv \partial_\theta W$. Due to scalar curvature, $R$ takes a nonzero value at the horizon. Then this region can be approximated by some sort of constant-curvature space. However, the exact results for such a black hole show that the mass parameter in the solution enters only in the combination $(\mu^2 - R/6)$ [3, 32]. Therefore, inserting the covariant metric into Eq. (3.2), we arrive at

$$(\partial_r W(r, \theta))^2 = -\frac{g_{rr}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}\left[(E - \Omega m)^2 + \left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\right.$$
$$\left.\times \left(\frac{m^2}{g_{\varphi\varphi}} + \frac{p_\theta^2}{g_{\theta\theta}} + M^2(r, \theta)\right)\right] \tag{3.3}$$

where $\Omega \equiv -g_{t\varphi}/g_{\varphi\varphi}$ and $M^2(r, \theta) \equiv \mu^2 - (\frac{1}{6} - \xi)R$. Equation (3.3) shows that $W(r, \theta)$ can be expressed as

$$W(r, \theta) = \pm\int^r \sqrt{\frac{-g_{rr}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}}\, K(r, \theta)\, dr + c(\theta) \tag{3.4}$$

where

$$K(r, \theta) = \sqrt{(E - \Omega m)^2 + \left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\left(\frac{m^2}{g_{\varphi\varphi}} + \frac{p_\theta^2}{g_{\theta\theta}} + M^2(r, \theta)\right)}$$

Consequently, the function $\psi(r, \theta)$ can be expressed as

$$\psi(r, \theta) = \exp\left[i\int^r \sqrt{\frac{-g_{rr}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}}\, K(r, \theta)\, dr\right]$$
$$+ A\exp\left[-i\int^r \sqrt{\frac{-g_{rr}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}}\, K(r, \theta)\, dr\right] \tag{3.5}$$

The constant $A$ is to be determined with the boundary condition. In Eq. (3.5) the amplitude is a slowly vary function and can be omitted. At a boundary $\Sigma_h$ at a small distance $h$ from the horizon $\Sigma$, the Solodukhin scattering condition [3] is given by

$$(n^\mu \partial_\mu \phi - \xi k \phi)\big|_{\Sigma_h} = 0 \tag{3.6}$$

where $n^\mu$ is a vector normal to $\Sigma_h$ and $k$ is the extrinsic curvature [30, 31] of $\Sigma_h$. For the black hole, after setting $n_\mu = (0, \sqrt{g_{rr}}, 0, 0)$, we find on $\Sigma_h$ that $k$ can be written as [33]

$$k(\Sigma_h) \approx \left[\frac{1}{2\sqrt{g_{rr}}(g_{tt} - g_{t\varphi}^2/g_{\varphi\varphi})}\frac{\partial}{\partial r}\left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}\right)\right]_{\Sigma_h} \tag{3.7}$$

Substitution of Eqs. (3.5) and (3.7) into (3.6) gives

$$\left[\sqrt{\frac{-(g_{tt} - g_{t\varphi}^2/g_{\varphi\varphi})}{g_{rr}}}\frac{\partial\psi(r, \theta)}{\partial r} - \xi^*\psi(r, \theta)\right]_{\Sigma_h} = 0 \tag{3.8}$$

where $\xi^* \equiv 2\pi\xi/\beta_H$ and $\beta_H$ is the Hawking inverse temperature. Making use of Eq. (3.8) and another boundary condition $\phi = 0$ for $r = r_E$ ($r_E < r_{VLS}$, where $r_{VLS}$ is the position of the velocity of the light surface [20, 25]), we find

$$2 \int_{r_+ + h}^{r_E} \sqrt{\frac{-g_{rr} g_{\varphi\varphi}}{g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2}}\, K(r, \theta)\, dr$$
$$= \nu \eta(K) + \pi \nu + 2\pi n(E, m, p_\theta, \theta) \tag{3.9}$$

with

$$\eta(K) = \arctan\left( \frac{2K(h)\xi^*}{K(h)^2 - \xi^{*2}} \right)$$

We can separate $n(E, m, p_\theta, \theta)$ into two parts as $n(E, m, p_\theta, \theta) = n_0(E, m, p_\theta, \theta) + n_1(E, m, p_\theta, \theta)$. Solving Eq. (3.9) and introducing integration of $\theta$, we obtain

$$n_0(E, m, p_\theta) = \frac{1}{\pi} \int d\theta \int_{r_+ + h}^{r_E} \sqrt{\frac{-g_{rr} g_{\varphi\varphi}}{g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2}}\, K(r, \theta)\, dr$$
$$n_1(E, m, p_\theta) = \frac{-\nu}{2\pi} \int \left[ \arctan\left( \frac{2K(h)\xi^*}{K(h)^2 - \xi^{*2}} \right) + \pi \right]_{\Sigma_h} d\theta \tag{3.10}$$

A nonminimally coupled scalar in thermal equilibrium at temperature $1/\beta$ in the EMDA black hole will be dragged, and it is reasonable to assume that the scalar field is rotating with dragging velocity $\Omega_0 \sim \Omega$. The Helmholtz free energy (3.1) is given by

$$\beta F = -\beta \int dm \int dp_\theta \int$$
$$\times \frac{n_0(E + \Omega_0 m, m, p_\theta) + n_1(E + \Omega_0 m, m, p_\theta)}{e^{\beta E} - 1}\, dE \tag{3.11}$$

Now let us use the Pauli–Villars regularization scheme [18] by introducing five regulator fields $\{\phi_i,\ i = 1, \ldots, 5\}$ of different statistics with masses $\{\mu_i,\ i = 1, \ldots, 5\}$ dependent on the ultraviolet cutoff [18] and with the same nonminimal coupling $\{\xi_i = \xi,\ i = 0, \ldots, 5\}$. If we rewrite the original scalar field $\phi = \phi_0$ and $\mu = \mu_0$, then these fields satisfy $\sum_{i=0}^{5} \Delta_i = 0$ and $\sum_{i=0}^{5} \Delta_i \mu_i^2 = 0$, where $\Delta_0 = \Delta_3 = \Delta_4 = +1$ for the commuting fields and $\Delta_1 = \Delta_2 = \Delta_5 = -1$ for the anticommuting fields. Since each of the fields makes a contribution to the Helmholtz free energy as in Eq. (3.11), the total Helmholtz free energy becomes $\beta \bar{F} = \sum_{i=0}^{5} \Delta_i \beta F_i = \beta \bar{F}_0 + \beta \bar{F}_1$. Substituting the EMDA black hole metric (2.1) of Ref. 29 and taking the integration of the $m$, $p_\theta$, and $E$, we find

$$\bar{F}_0 = \frac{-1}{48}\frac{\beta_H}{\beta^2}\int d\theta V \sin\theta \sum_{i=0}^{5} \Delta_i M_i^2(r_+, \theta) \ln M_i^2(r_+, \theta)$$
$$- \frac{1}{2880}\frac{\beta_H^3}{\beta^4}\int d\theta V \sin\theta$$
$$\times \left\{\frac{-32Mr_+^3 + 4D(8r_+ - 3D)(r_+^2 - 2Dr_+ + a^2)}{\Delta^3}\right.$$
$$+ \frac{4[2M(3r_+ - 2D) - D^2]}{\Delta^2} + \frac{4MD(\Delta - 2Dr_+)}{\Delta^3}$$
$$\left. + \frac{2a^2(1 + \cos^2\theta)\Delta - 16Ma^2 r_+ \cos^2\theta}{\Delta^3}\right\} \sum_{i=0}^{5} \Delta_i \ln M_i^2 \tag{3.12}$$

$$\bar{F}_1 = -\frac{|\xi|}{4\beta}\int d\theta V \sin\theta \sum_{i=0}^{5} \Delta_i M_i^2 \ln M_i^2 \tag{3.13}$$

where $V = r_+^2 - 2Dr_+ + a^2$, $\Delta = r_+^2 - 2Dr_+ + a^2 \cos^2\theta$, and $M_i^2 = \mu_i^2 - (\frac{1}{6} - \xi)\, R(r_+, \theta)$. Using the formula $S = [\beta^2 \partial F/\partial\beta]_{\beta=\beta_H}$ and the assumption that the scalar curvature at the horizon is much smaller than each $\mu_i$, and integrating over $\theta$, we obtain the expression for the statistical mechanical entropy

$$S^{SM} = \frac{A_\Sigma}{48\pi}(1 + 6|\xi|)\sum_{i=0}^{5}\Delta_i \mu_i^2 \ln \mu_i^2$$
$$- \frac{1}{2}\left\{\frac{D^2}{2}\left(\frac{1}{6} - \xi\right)^2\left[\frac{Mr_+ + a^2}{(2Mr_+ - a^2)^2} - \frac{2Mr_+(Mr_+ - 2a^2)}{a(2Mr_+ - a^2)^{5/2}}\right.\right.$$
$$\left.\times \tan^{-1}\frac{a}{\sqrt{2Mr_+ - a^2}}\right] - \frac{1}{180}\left[\frac{DM(-7D + 2r_+)}{r_+(r_+ - 2D)^2} - \frac{D^2}{r_+^2 - 2Dr_+}\right.$$
$$\left. + \frac{2DM^2(7D - 2r_+)}{a(r_+ - 2D)^2\sqrt{r_+^2 - 2Dr_+}}\tan^{-1}\left(\frac{a}{\sqrt{r_+^2 - 2Dr_+}}\right)\right]$$
$$+ \frac{1}{90}\left[\frac{D(-4D^2 - 7DM + 2Dr_+ + 8Mr_+)}{2r_+(r_+ - 2D)^2} + \frac{3D^2 - 8Dr_+ + 4r_+^2}{2r_+^2 - 4Dr_+}\right.$$
$$+ \frac{DM(4D^2 + 7DM - 2Dr_+ - 8Mr_+)}{a(r_+ - 2D)\sqrt{r_+^2 - 2Dr_+}}$$

$$\times \tan^{-1}\left(\frac{a}{\sqrt{r_+^2 - 2Dr_+}}\right)\Bigg]\Bigg\}\sum_{i=0}^{5}\Delta_i \ln \mu_i^2 \tag{3.14}$$

The first part in Eq. (3.14) has a geometric character. The second part is not proportional to the horizon area and depends on the black hole characteristics ($M$, $Q$, and $J$).

## 4. THERMODYNAMIC ENTROPY OF THE EMDA BLACK HOLE

Following Mann and Solodukin [16], we will first show that a Euclidean manifold obtained by Wick rotation of the EMDA black hole has a conical singularity. By using the conical singularities method we then obtain the tree-level thermodynamic entropy and its one-loop quantum corrections for the EMDA black hole due to a minimally coupled scalar field.

For the black hole (2.1), a pair of Killing vectors is given by

$$K = \left(1, 0, 0, \frac{a}{r^2 + a^2 - 2Dr}\right), \qquad \tilde{K} = (a \sin^2\theta, 0, 0, 1). \tag{4.1}$$

The one-forms dual to $K$ and $\tilde{K}$ are

$$\omega = \left(\frac{r^2 + a^2 - 2Dr}{\Delta}, 0, 0, -\frac{(r^2 + a^2 - 2Dr)a \sin^2\theta}{\Delta}\right)$$

$$\tilde{\omega} = \left(-\frac{a}{\Delta}, 0, 0, \frac{r^2 + a^2 - 2Dr}{\Delta}\right) \tag{4.2}$$

Then, the metric (2.1) can be written as

$$ds^2 = \frac{\Sigma\Delta}{(r^2 + a^2 - 2Dr)}\,\omega^2 + \frac{\Delta}{\Sigma}\,dr^2 + \Delta(d\theta^2 + \sin^2\theta\tilde{\omega}^2) \tag{4.3}$$

Euclideanize the metric by setting $t = i\tau$ and $a = i\hat{a}$; then the Euclidean vectors (4.1) and the corresponding one-forms (4.2) take the form

$$K = \left(1, 0, 0, -\frac{\hat{a}}{r^2 - \hat{a}^2 - 2Dr}\right), \qquad \tilde{K} = (\tilde{a} \sin^2\theta, 0, 0, 1) \tag{4.4}$$

$$\omega = \left(\frac{r^2 - \hat{a}^2 - 2Dr}{\hat{\Delta}}, 0, 0, -\frac{(r^2 - \hat{a}^2 - 2Dr)\hat{a} \sin^2\theta}{\hat{\Delta}}\right)$$

$$\tilde{\omega} = \left(\frac{\hat{a}}{\hat{\Delta}}, 0, 0, \frac{r^2 - \hat{a}^2 - 2Dr}{\hat{\Delta}}\right) \tag{4.5}$$

The metric (4.3) is changed to

$$ds^2 = \frac{\hat{\Sigma}\hat{\Delta}}{(r^2 - \hat{a}^2 - 2Dr)} \omega^2 + \frac{\hat{\Delta}}{\hat{\Sigma}} dr^2 + \hat{\Delta}(d\theta^2 + \sin^2\theta\tilde{\omega}^2) \tag{4.6}$$

It is useful to introduce a new radial variable

$$\tilde{\Sigma} = \gamma(r - \hat{r}_+) = \gamma^2 \frac{x^2}{4}, \qquad \gamma = 2\sqrt{\left(M - \frac{Q^2}{2Me^{2\Phi_0}}\right)^2 - a^2} \tag{4.7}$$

Up to terms $o(x^2)$ the metric (4.6) becomes

$$ds^2 = \hat{\Delta} ds_{c2}^2 + ds_{\Sigma}^2 \tag{4.8}$$

with

$$ds_{c2}^2 = dx^2 + \frac{\gamma^2 x^2}{4(\hat{r}_+^2 - a^2 - 2D\hat{r}_+)^2} \omega^2 \tag{4.9}$$

$$ds_{\Sigma}^2 = \hat{\Delta}\,(d\theta^2 + \sin^2\theta\tilde{\omega}^2) \tag{4.10}$$

Introducing the new angle coordinate

$$d\chi = \frac{\hat{r}_+^2 - a^2 - 2D\hat{r}_+}{\beta\tilde{\Delta}} (d\tau - \hat{a}\sin^2\theta\, d\varphi)$$

we find that Eq. (4.9) reads

$$ds_{c2\alpha}^2 = dx^2 + \alpha^2 x^2 d\chi^2 \tag{4.11}$$

where $\alpha = \beta/\beta_H$. Equations (4.10), (4.8), and (4.11) take a similar form to Eqs. (2.8), (2.9), and (3.1) in ref. 16, respectively. Therefore, the discussions in ref. 16 are valid for the EMDA black hole (2.1).

From the Euclidean metric (4.6) we can define a pair of vectors orthogonal to the horizon,

$$\begin{aligned} n_1^\mu &= \left(0, \sqrt{\frac{\hat{\Sigma}}{\hat{\Delta}}}, 0, 0\right) \\ n_2^\mu &= \left(\frac{r^2 - a^2 - 2Dr}{\sqrt{\hat{\Sigma}\hat{\Delta}}}, 0, 0, -\frac{\hat{a}}{\sqrt{\hat{\Sigma}\hat{\Delta}}}\right) \\ n_\mu^1 &= \left(0, \sqrt{\frac{\hat{\Delta}}{\hat{\Sigma}}}, 0, 0\right) \end{aligned} \tag{4.12}$$

$$n_\mu^2 = \left(\sqrt{\frac{\hat{\hat{\Sigma}}}{\hat{\Delta}}},\ 0,\ 0,\ -\sqrt{\frac{\hat{\hat{\Sigma}}}{\hat{\Delta}}}\sin^2\theta\right) \tag{4.13}$$

For the EMDA black hole we have

$$(K^a K^a)_{r_H} = 0$$

$$[\mathrm{tr}(K\cdot K)]_{r_H} = (K^a_{\mu\nu}\, K^{\mu\nu}_a)_{r_H} = 0 \tag{4.14}$$

where $K^a_{\mu\nu} = -\gamma^\alpha_\mu\, \gamma^\beta_\nu\, \nabla\, \alpha\, n^a_\beta$ is the extrinsic curvature, and $K^a = g^{\mu\nu}\, K^a_{\mu\nu}$ is the trace of the extrinsic curvature [16].

Following the discussion in Solodukhin [11] and Mann and Solodukhin [16], we obtain the tree-level thermodynamic entropy $S^{TD}(G_B, c^i_B)$ and its one-loop quantum corrections $S^{TD}_{div}$ for the EMDA black hole, respectively, as

$$\begin{aligned}
&S^{TD}(G_B, c^i_B)\\
&= \frac{A_\Sigma}{4G_B} - 8\pi \int d\theta\, d\varphi \left[\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\left(c^1_B R + \frac{c^2_B}{2}\sum_{a=1}^{2} R_{\mu\nu}n^\mu_i n^\nu_i c^3_B\right.\right.\\
&\quad \left.\left.\times \sum_{a,b=1}^{2} R_{\mu\nu\alpha\beta}n^\mu_i n^\nu_j n^\alpha_i n^\beta_j\right)\right]_{r_H}\\
&= \frac{A_\Sigma}{4G_B} - 16\pi^2 c^1_B D^2\left[\frac{Mr_+ + a^2}{(2Mr_+ - a^2)^2} - \frac{2Mr_+(Mr_+ - 2a^2)}{a(2Mr_+ - a^2)^{5/2}}\right.\\
&\quad \left.\times \tan^{-1}\frac{a}{\sqrt{2Mr_+ - a^2}}\right] - 8\pi^2 c^2_B\left[\frac{DM(-7D + 2r_+)}{r_+(r_+ - 2D)^2} - \frac{D^2}{r_+^2 - 2Dr_+}\right.\\
&\quad \left.+ \frac{2DM^2(7D - 2r_+)}{a(r_+ - 2D)^2\sqrt{r_+^2 - 2Dr_+}}\tan^{-1}\left(\frac{a}{\sqrt{r_+^2 - 2Dr_+}}\right)\right]\\
&\quad - 16\pi^2 c^3_B\left[\frac{D(-4D^2 - 7DM + 2Dr_+ + 8Mr_+)}{2r_+(r_+ - 2D)^2} + \frac{3D^2 - 8Dr_+ + 4r_+^2}{2r_+^2 - 4Dr_+}\right.\\
&\quad \left.+ \frac{DM(4D^2 + 7DM - 2Dr_+ - 8Mr_+)}{a(r_+ - 2D)\sqrt{r_+^2 - 2Dr_+}}\tan^{-1}\left(\frac{a}{\sqrt{r_+^2 - 2Dr_+}}\right)\right]
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&S^{TD}_{div}\\
&= \frac{A_\Sigma}{48\pi\epsilon^2}(1 - 6\xi) + \left\{\frac{D^2}{2}\left(\frac{1}{6} - \xi\right)^2\left[\frac{Mr_+ + a^2}{(2Mr_+ - a^2)^2}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2Mr_+(Mr_+ - 2a^2)}{a(2Mr_+ - a^2)^{5/2}} \tan^{-1} \frac{a}{\sqrt{2Mr_+ - a^2}}\Bigg] \\
&- \frac{1}{180}\left[\frac{DM(-7D + 2r_+)}{r_+(r_+ - 2D)^2} - \frac{D^2}{r_+^2 - 2Dr_+}\right. \\
&\left. + \frac{2DM^2(7D - 2r_+)}{a(r_+ - 2D)^2\sqrt{r_+^2 - 2Dr_+}} \tan^{-1}\left(\frac{a}{\sqrt{r_+^2 - 2Dr_+}}\right)\right] \\
&+ \frac{1}{90}\left[\frac{D(-4D^2 - 7DM + 2Dr_+ + 8Mr_+)}{2r_+(r_+ - 2D)^2} + \frac{3D^2 - 8Dr_+ + 4r_+^2}{2r_+^2 - 4Dr_+}\right. \\
&\left.\left. + \frac{DM(4D^2 + 7DM - 2Dr_+ - 8Mr_+)}{a(r_+ - 2D)\sqrt{r_+^2 - 2Dr_+}} \tan^{-1}\left(\frac{a}{\sqrt{r_+^2 - 2Dr_+}}\right)\right]\right\} \ln \frac{L}{\epsilon}
\end{aligned} \tag{4.16}
$$

where $G_B$, $c_B^i$ $(i = 1, 2, 3)$ represent bare constants (tree-level). The expressions (4.14)–(4.16) are restored to real values of the parameters $t$ and $a$. Noting that the Pauli–Villars regularization scheme causes a factor $-1/2$ in the second part in Eq. (3.14), we know that the statistical mechanical entropy (3.14) coincides with the one-loop quantum correction to thermodynamic entropy (4.16) for $\xi \leq 0$ coupling.

## 5. SUMMARY AND DISCUSSION

Combing the tree-level entropy (4.15) with the one-loop correction (4.16), we find that the divergence of the thermodynamic entropy can be absorbed in the renormalization of the coupling constants

$$
\begin{aligned}
\frac{1}{G_{ren}} &= \frac{1}{G_B} + \frac{1}{2\pi\epsilon^2}\left(\frac{1}{6} - \xi\right) \\
c_{ren}^1 &= c_B^1 - \frac{1}{32\pi^2}\left(\frac{1}{6} - \xi\right)^2 \ln \frac{L}{\epsilon} \\
c_{ren}^2 &= c_B^2 + \frac{1}{32\pi^2}\frac{1}{90} \ln \frac{L}{\epsilon} \\
c_{ren}^3 &= c_B^3 - \frac{1}{32\pi^2}\frac{1}{90} \ln \frac{L}{\epsilon}
\end{aligned} \tag{5.1}
$$

Since we considered the case that terms quadratic in curvature are preserved in the renormalized action, the black hole entropy can be expressed as [14–34]

$$S^{BH}\,(G_{ren},\, c^i_{ren}) = S^{TD}\,(G_{ren},\, c^i_{ren})$$
$$S^{BH}\,(G_B,\, c^i_B) = S^{TD}\,(G_B,\, c^i_B) \tag{5.2}$$

Therefore, we obtain for $\xi \leq 0$ the relation

$$S^{BH}\,(G_{ren},\, c^i_{ren}) = S^{BH}\,(G_B,\, c^i_B) + S^{SM} \tag{5.3}$$

which agrees the static black hole results shown in refs. 10, 15, 18, 35, and 36. Equation (5.3) shows that the presence of the bare pure geometrical contribution $S^{BH}\,(G_B,\, c^i_B)$ excludes the possibility of identifying $S^{BH}(G_{ren},\, c^i_{ren})$ with $S^{SM}$ if we use standard renormalization approach.

The statistical mechanical and thermodynamic entropies for some well-known black holes, such as Garfinkle–Horowitz–Strominger dilaton black hole (for the case $a \to 0$, $\xi = 0$) [37] and the Kerr black hole ($D = 0$), are special cases of the results (3.14) and (4.16). It is interesting to note that for the Kerr black hole, both the statistical mechanical entropy and one-loop correction to the thermodynamic entropy are given by

$$S_{div} = \frac{A_\Sigma}{48\pi\epsilon^2} + \frac{1}{45}\ln\frac{L}{\epsilon}$$

To conclude, the statistical mechanical entropy due to nonminimally coupled scalar fields in the EMDA black hole has been studied by a "brick wall" model in which the original Dirichlet condition is replaced by a scattering ansatz for the field functions at the event horizon and with the Pauli–Villars regularization scheme. Its thermodynamic entropy was also investigated by using the conical singularities method. Comparing the statistical mechanical and thermodynamic entropies and noting that the Pauli–Villars regularization scheme causes a factor $-1/2$ in the second part in Eq. (3.14), we showed that, for the EMDA black hole, the statistical mechanical entropy and the one-loop correction to the thermodynamic entropy are equivalent for the coupling $\xi \leq 0$. From the tree-level entropy (4.15) and one-loop correction (4.16), we found that the divergence can be absorbed in the renormalization of the coupling constants. A relation between the statistical mechanical entropy of quantum excitations of the EMDA black hole and its thermodynamic entropy for the case $\xi \leq 0$ was obtained. It shows that, if we use the standard scheme, the presence of the bare pure geometrical contribution excludes the possibility of identifying the statistical mechanical entropy with the thermodynamic entropy.

## ACKNOWLEDGMENTS

This work was partially supported by the National Nature Science Foundation of China under Grant No. 19975018 and the Nature Science Foundation of Hunan Province.

## REFERENCES

1. V. P. Frolov and D. V. Fursaev, *Class. Quantum Grav.* **15**, 2041 (1998).
2. S. N. Solodukhin, *Phys. Rev. D* **54**, 3900 (1996).
3. S. N. Solodukhin, *Phys. Rev. D* **56**, 4968 (1997).
4. V. P. Frolov, D. V. Fursaev, and A. I. Zelnikov, *Phys. Lett. B* **382**, 220 (1996).
5. V. P. Frolov, D. V. Fursaev, and A. I. Zelnikov, *Phys. Rev. D* **54**, 2711 (1996).
6. D. V. Fursaev, Euclidean and canonical formulations of statistical mechanics in the presence of Killing horizon, hep-th/9709213.
7. S. N. Solodukhin, *Phys. Rev. D* **51**, 618 (1995).
8. D. V. Fursaev, *Class. Quantum Grav.* **11**, 1431 (1994).
9. G. Cognola, K. Kirsten, and L. Vanzo, *Phys. Rev. D* **49**, 1029 (1994).
10. S. N. Solodukhin, *Phys. Rev. D* **51**, 609 (1995).
11. S. N. Solodukhin, *Phys. Rev. D* **52**, 7046 (1995).
12. A. O. Barvinsky and S. N. Solodukhin, *Nucl. Phys. B* **479**, 305 (1996).
13. D. V. Fursaev, *Phys. Lett. B* **334**, 53 (1994).
14. D. V. Fursaev, *Mod. Phys. Lett. A* **10**, 649 (1995).
15. D. V. Fursaev and S. N. Solodukhin, *Phys. Lett. B* **365**, 51 (1996).
16. R. B. Mann and S. N. Solodukhin, *Phys. Rev. D* **54**, 3932 (1996).
17. G. 't Hooft, *Nucl. Phys. B* **256**, 727 (1985).
18. J. G. Demers, R. Lafrance, and R. C. Myers, *Phys. Rev. D* **52**, 2245 (1995).
19. A. Ghosh and P. Mitra, *Phys. Rev. Lett.* **73**, 2521 (1994).
20. M. H. Lee and J. K. Kim, *Phys. Lett. A* **212**, 323 (1996).
21. R. B. Mann, L. Tarasov, and A. Zelnikov, *Class. Quantum Grav.* **9**, 1487 (1992).
22. A. Romeo, *Class. Quantum Grav.* **13**, 2797 (1996).
23. S. P. Kim *et al.*, *Phys. Rev. D* **55**, 2159 (1997).
24. G. Cognola and P. Lecca, *Phys. Rev. D* **57**, 2159 (1998).
25. M. H. Lee and J. K. Kim, *Phys. Rev. D* **54**, 3904 (1996).
26. F. Belgiorno and S. Liberati, *Phys. Rev. D* **53**, 3172 (1996).
27. Jiliang Jing, *Int. J. Theor. Phys.* **37**, 1441 (1998).
28. Jiliang Jing, *Chin. Phys. Lett.* **14**, 495 (1997).
29. A. Garcia, D. Galtsov, and O. Kechkin, *Phys. Rev. Lett.* **74**, 1276 (1995).
30. R. M. Wald, *General Relativity,* Cambridge University Press, Cambridge (1984).
31. S. W. Hawking, and G. F. R. Eillis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge (1973).
32. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge (1982).
33. Jiliang Jing, *Nucl. Phys. B* **476**, 548 (1996).
34. T. A. Jacobson, G. Kang, and R. C. Myers, *Phys. Rev. D* **49**, 6587 (1994).
35. F. Larsen and F. Wilczek, *Nucl. Phys. B* **458**, 249 (1996).
36. D. Kabat, *Nucl. Phys. B* **453**, 281 (1995).
37. D. Garfinkle, G. T. Horowitz, and A. Strominger, *Phys. Rev. D* **43**, 3140 (1991).